A Gaussian mixture model of \( r \) components consists of \( r \) Gaussian distributions, \( N(\mu_i, \Sigma_i) \) for \( i = 1, \ldots, r \), where \( \mu_i \in \mathbb{R}^d \) is the expectation (or mean) and \( \Sigma_i \in \mathbb{R}^{d \times d} \) is the covariance matrix. Each sample of the distribution is drawn from one Gaussian component and the one that is sampled is drawn from the \( i \)-th Gaussian distribution is \( \omega_i > 0 \). The density function of the Gaussian mixture model is the weighted sum of the \( r \) density functions of component Gaussian distributions. Learning a Gaussian mixture model is to estimate parameters of the model \( \omega_i, \mu_i, \Sigma_i \) from given samples of the model. We proposed an algorithm\(^\text{[1]}\) to learn Gaussian mixture models by using tensor decompositions.

Tensor Decompositions with Incomplete Entries

The tensor \( \mathcal{F} \) is rewritten as

\[
\mathcal{F} = \lambda_1 \left( \begin{smallmatrix} 1 \\ u_{11} \\ \vdots \\ u_{1d} \end{smallmatrix} \right) \otimes \cdots \otimes \lambda_r \left( \begin{smallmatrix} 1 \\ u_{r1} \\ \vdots \\ u_{rd} \end{smallmatrix} \right),
\]

where \( \lambda_i = \omega_i(\mu_j | \mu_1) + \omega_i(\mu_j | \mu_2) \in \mathbb{R}^c \). In the following, we discuss how to find the decomposition of \( \mathcal{F} \) from partial entries (\( \mathcal{F}_{ijkl} \)).

- When \( r < \frac{d}{k} - 1 \), there is a unique generalizing matrix \( G = (G_{r+1,k+1})_{1 \leq r,k \leq d-1} \) of \( \mathcal{F} \) such that

\[
\sum_{r=1}^{d-1} \sum_{k=1}^{d-1} G(k,r+1)F_{ijkl} - F_{ijkl} = 0, \quad l = 0, \ldots, d-1,
\]

for all \( l \leq r < j \leq d-1 \) and \( 0 \leq k \leq d-1 \). By choosing \( r > l + 1 \) and \( j \), entries \( \mathcal{F}_{ijkl}, F_{ijkl} \) in the above equations are parts of \( \mathcal{F}_{ikhj} \) and hence are known. Thus, the matrix \( G \) can be found by solving the above linear equations.

- For \( r = l+1, \ldots, d-1 \), it holds that

\[
\Omega_{ijl} = \left( \begin{smallmatrix} \omega_i(\mu_j | \mu_1) \\ \omega_i(\mu_j | \mu_2) \end{smallmatrix} \right) \text{where}, \quad N_i(\mathcal{G}) = \left( \begin{smallmatrix} G(1,r+1) & \cdots & G(r,r+1) \\ G(1,r+1) & \cdots & G(r,r+1) \end{smallmatrix} \right).
\]

The above equations illustrate that vectors \( (u_{1j}), \ldots, (u_{rj}) \) are common eigenvectors of \( N_{ij} \). Thus, vectors \( u_{1j}, \ldots, u_{rj} \) can be recovered by finding common eigenvectors and corresponding eigenvalues of \( N_{ij} \). Finally, scalars \( \lambda_i \) can be obtained by solving a linear system.

Learning Diagonal Gaussian Mixture Models

For simplicity, we assume that moments \( M_1 = \mathbb{E}[z] \) and \( M_2 = \mathbb{E}[x \times z] \) are given exactly.

- The decomposition of \( \mathcal{F} \) can be recovered from \( \mathcal{F}_{ijkl} = \mathcal{F}_{ijko} \) by using incomplete tensor decomposition. The decomposition can be recovered as \( \mathcal{F} = \sum_{p=1}^{d-1} \mathcal{F}_{ijkl}^{(p)} \), where \( \mathcal{F}_{ijkl}^{(p)} = \langle z_{ijkl} \rangle \) for all \( (i,j,k) \in \Omega \).

In the definition, \( \mathcal{F} \) has the decomposition \( \mathcal{F} = \sum_{p=1}^{d-1} \mathcal{F}_{ijkl}^{(p)} \) which is unique when \( p \) is small. Thus, the recovered decomposition can be used to estimate \( \omega_i, \mu_i, \Sigma_i \) of the Gaussian mixture model and then covariance matrices \( \Sigma_i \) can be found by considering \( M_2 = \mathcal{F} \).

Table 1: Classification accuracy on textures

<table>
<thead>
<tr>
<th>Texture</th>
<th>Ours</th>
<th>EM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bark.0001</td>
<td>0.9841</td>
<td>0.9068</td>
</tr>
<tr>
<td>Grass.0001</td>
<td>0.9046</td>
<td>0.8854</td>
</tr>
<tr>
<td>Brick.0004</td>
<td>0.9220</td>
<td>0.9048</td>
</tr>
<tr>
<td>Fabric.0013</td>
<td>0.8376</td>
<td>0.8413</td>
</tr>
<tr>
<td>Tiling.0005</td>
<td>0.5107</td>
<td>0.7150</td>
</tr>
</tbody>
</table>

\[\text{[1]}\] Bingxi Guo, Jiawang Nie, and Zi Yang. "Learning diagonal Gaussian mixture models and incomplete tensor decompositions." In preparation.