Robust Data-driven PDE Identification from Single Noisy Trajectory

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Contribution: Overcome Heavy Data Noise

Data-driven PDE Identification aims at automatic PDE modeling based on experimental data. As differential operators are unbounded, this inverse procedure is susceptible to noise. We propose an effective denoising technique (**SDD**) and two model selection schemes (**ST** and **SC**) to greatly improve the stability and precision.

Problem Overview

Given a dataset U sampled from a *single* solution $u : [0,T) \times \mathbb{R}^D \to \mathbb{R}$ of an evolutionary PDE

$$u_t = \mathcal{F}^*(u) \ ,$$

with an unknown differential operator $\mathcal{F}^*$, the goal is to find an operator $\widehat{\mathcal{F}}$ based on U such that

$$\widehat{\mathcal{F}} \approx \mathcal{F}^* \ .$$

Here $T > 0$ is the time limit of the observation; D is the spacial dimension; and the data is noisy:

$$U_i^n = u(\mathbf{x}_i, t^n) + \varepsilon_i^n,\ \varepsilon_i^n \overset{\text{i.i.d.}}{\sim} \text{Normal}(0, p\%\|u\|_2) \ .$$

In this work, we assume that $\mathcal{F}^*$ is in an algebra of polynomial differential operators over $\mathbb{R}$, i.e.,

$$\mathcal{F}^*(u) = c_0 + c_1 u + c_2 u_x + \cdots + c_m u u_x + \cdots$$

where $c_1, c_2, \cdots \in \mathbb{R}$ can be mostly 0, and each monomial is a *feature variable*. The data-driven PDE identification is closely related to a *sparse regression* or *dictionary learning* problem:

$$\min \|\mathbf{c}\|_0 \ , \quad \text{subject to } \|F\mathbf{c} - D_t U\|_2^2 \le \varepsilon \ ,$$

where $\varepsilon > 0$, $\mathbf{c}^T = (c_0, c_1, \dots)$ is the coefficient vector, F is the *feature matrix* whose columns are discrete approximations of the feature variables, and $D_t U$ is a finite difference estimation of u_t.

References

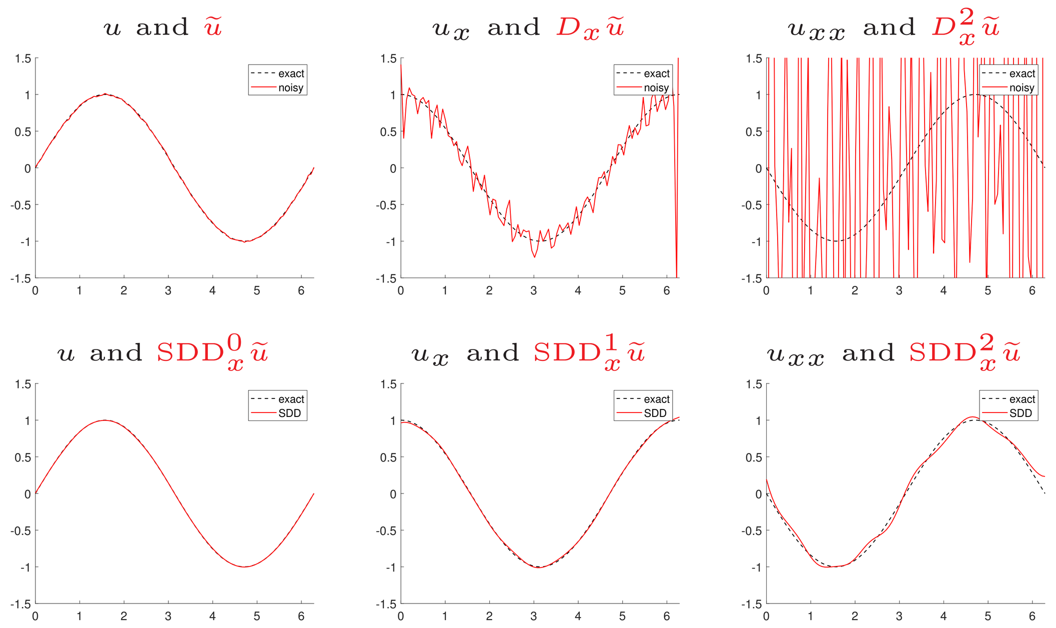
[1] W. Dai, O. Milenkovic Subspace pursuit for compressive sensing signal reconstruction IEEE transaction on Information Theory, 2009.

[2] H. Schaeffer, Learning partial differential equations via data discovery and sparse optimization In *Royal Society A: Mathematical, Physical and Engineering Sciences*, 2017.

[3] S.H. Kang, W. Liao, Y. Liu IDENT: Identifying differential equations with numerical time evolution arXiv preprint arXiv:1904.03538, 2019. (Submitted)

[4] Y. H, S.H. Kang, W. Liao, H. Liu, Y. Liu Robust PDE identification from noisy data arXiv preprint arXiv:2006.06557, 2020. (Submitted)

Noise Suppression by SDD



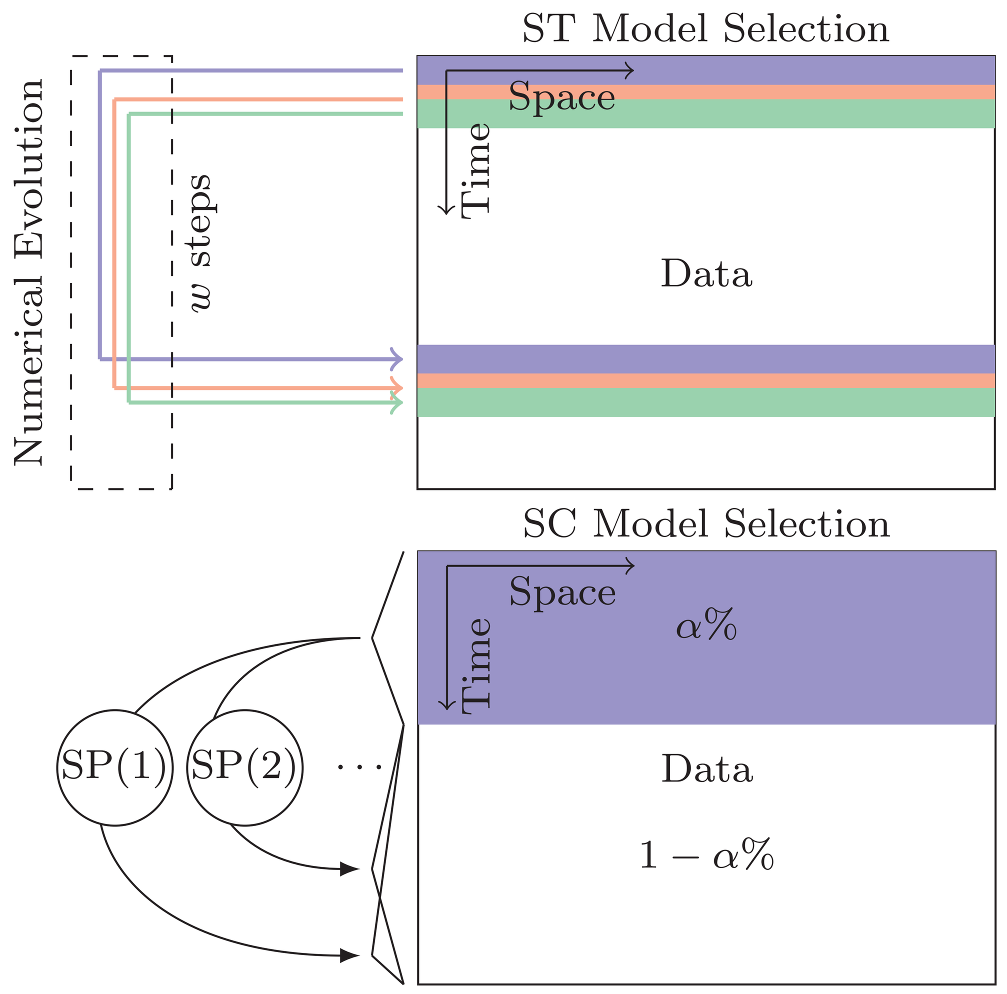
Successively Denoised Differentiation (SDD) effectively reduces noise:

$$\text{SDD}_x^l U := (\mathcal{S}D_x)^l \mathcal{S}U \ , \quad \text{for integer } l \ge 0 \ ,$$

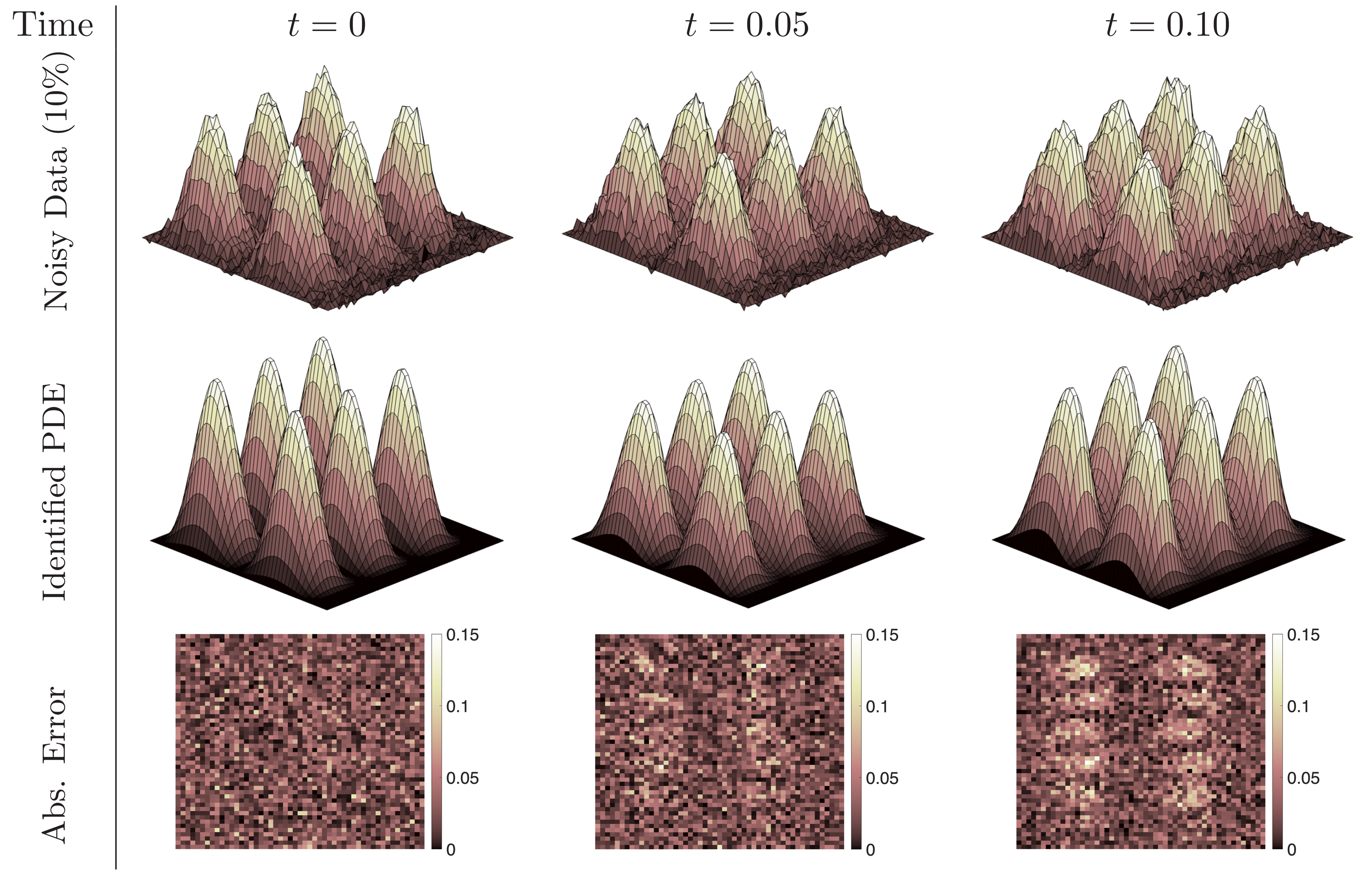
approximates the l-th order derivative of u. $\mathcal{S}$ is a smooth operator, and D_x is a finite difference scheme.

ST and SC Model Selection

Subspace pursuit (SP) [1] is a sparse algorithm which allows direct control of the ℓ_0-norm of the solution. We propose **Subspace Pursuit Time Evolution (ST)** and **Subspace Pursuit Cross Validation (SC)**.

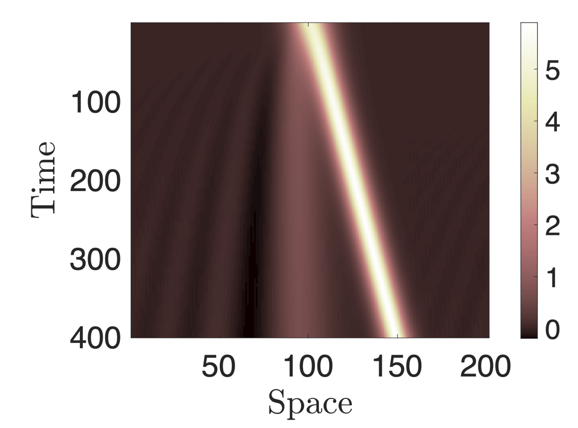


An Example of PDE Identification in 2D



Our methods **handle high level of noise**. From the noisy data (10%) generated by a single trajectory of $u_t = 0.02u_{xx} - uu_y$, both ST and SC identify the correct feature variables: u_{xx} and uu_y. The identified model is $u_t = 0.0134u_{xx} - 0.8675uu_y$, and the simulation errors are relatively small.

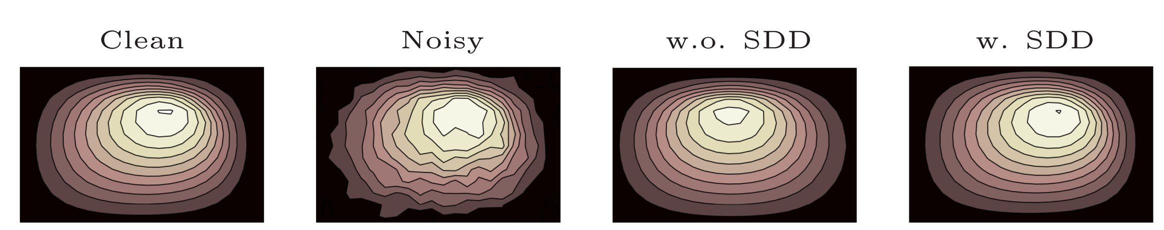
KdV Equation



The true PDE takes the form $u_t = -6uu_x - u_{xxx}$. From a dictionary of **15** terms: $1, u_x, u_{xx}, u_{xxx}$ and their pairwise products, we identify the model $u_t = -6.135uu_x - 1.0580u_{xxx}$.

Necessity of SDD

SDD is important for correct identification. The true PDE is $u_t = -0.3u_x - 0.5uu_x - 0.5uu_y$; without SDD, the identified model is $u_t = -0.2140u_x + 0.0074u_{yy} - 0.6533uu_x$; and with SDD, we identify $u_t = -0.2599u_x - 0.5513u_y - 0.4434uu_y$. We visualize the model difference by showing the model simulation.



For more examples, please refer to [4].

Some Comparison

Our methods are **free from post-thresholding**.

True PDE	$u_t = -uu_x$ $0 \le x \le 1 \ , \ 0 < t \le 0.05$
Method	0% **noise**
[2]	$u_t = -0.95uu_x - 0.01u + \cdots$
ST(20), SC(1/200)	$u_t = -1.0013uu_x$
	1% **noise**
[2]	$u_t = -0.89uu_x - 0.13u + \cdots$
ST(20), SC(1/200)	$u_t = -0.97uu_x$
	5% **noise**
[2]	$u_t = -0.35uu_x + 0.09u^2 + \cdots$
ST(20), SC(1/200)	$u_t = -0.98uu_x$

Our methods are **more robust against heavy noise** compared to [3] using the following metrics: