Reduced training data for dynamical systems

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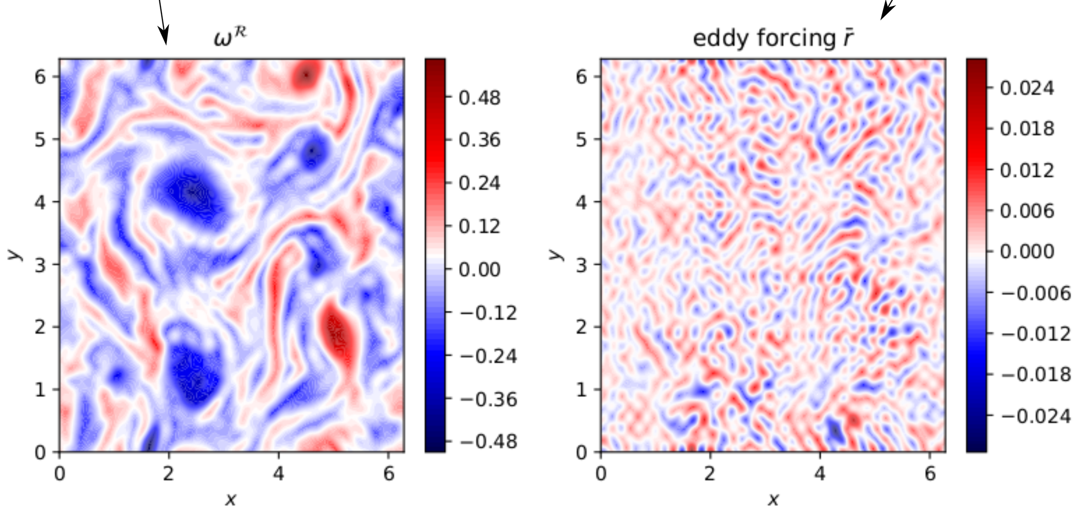
1) Multiscale governing equations

- Forced dissipative vorticity eqs, coarse-grained to 64×64 grid
- Coarse graining: subgrid-scale (SGS) model now required

$$\frac{\partial \bar{\omega}}{\partial t} + \overline{J\left(\bar{\psi}, \bar{\omega}\right)} = \nu \nabla^2 \bar{\omega} + \mu \left(F - \bar{\omega}\right) - \boxed{\textit{SGS model}},$$

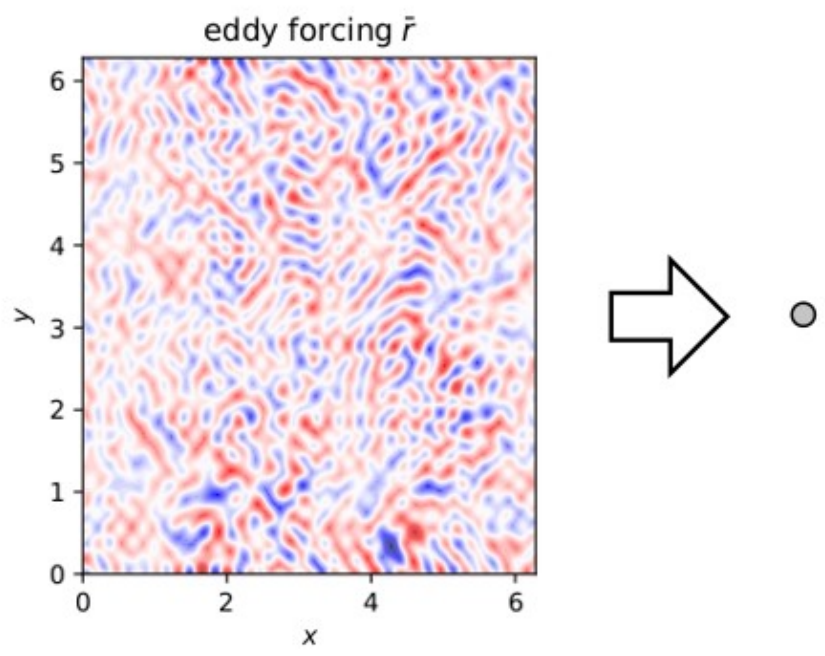
$$\nabla^2 \bar{\psi} = \boxed{\bar{\omega}}.$$



2) Goal

- Goal: predict **statistics of time series**, e.g. energy $E(t)$
- How: machine learning of a SGS surrogate, using data from 256×256 simulation
- Problem: exact SGS is **full-field** $\rightarrow$ complex surrogate
- Our goal is not full field: **compress ML training data**

New SGS model where unclosed components are time series



3) Assumptions

- Our new SGS term has the following form:

$$\text{Reduced SGS} = \sum_{i=1}^{d} \tau_i(t) P_i(x, y, t),$$

- and must 'track' d time-dependent quantities of interest:

$$Q_i(t) = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} q_i(\bar{\omega}, \bar{\psi}; x, y, t)\, \mathrm{d}x\mathrm{d}y, \quad i = 1, \cdots, d.$$

such that $Q_i^{ref}(t) - Q_i(t)$ is small $\forall t$ in training period

4) Compute effect of assumptions

- Derive transport equation of the Q_i:

$$\frac{\mathrm{d}Q_i}{\mathrm{d}t} = \ldots + \left(\frac{\partial q_i}{\partial \bar{\omega}}, \textit{Reduced SGS}\right) = \ldots + \sum_{j=1}^{d} \tau_j \left(\frac{\partial q_i}{\partial \omega}, P_j\right)$$

- Every Q_i has d SGS terms: let's remove $\sum_{j=1}^{d}$
- Simplify, make P_j orthogonal:

$$P_i := \frac{\partial q_i}{\partial \bar{\omega}} - \sum_{j=2}^{d} c_{i,j}(t) \frac{\partial q_j}{\partial \bar{\omega}}$$

$$\boxed{\left(\frac{\partial q_i}{\partial \bar{\omega}}, P_j\right) = 0 \;\text{ if }\; i \neq j} \quad \forall t$$

5) Extract τ_i from data

- Due to orthogonality, transport equation of the Q_i become:

$$\frac{\mathrm{d}Q_i}{\mathrm{d}t} = \ldots + \left(\frac{\partial q_i}{\partial \bar{\omega}}, \textit{Reduced SGS}\right) = \ldots + \tau_i \left(\frac{\partial q_i}{\partial \omega}, P_i\right)$$

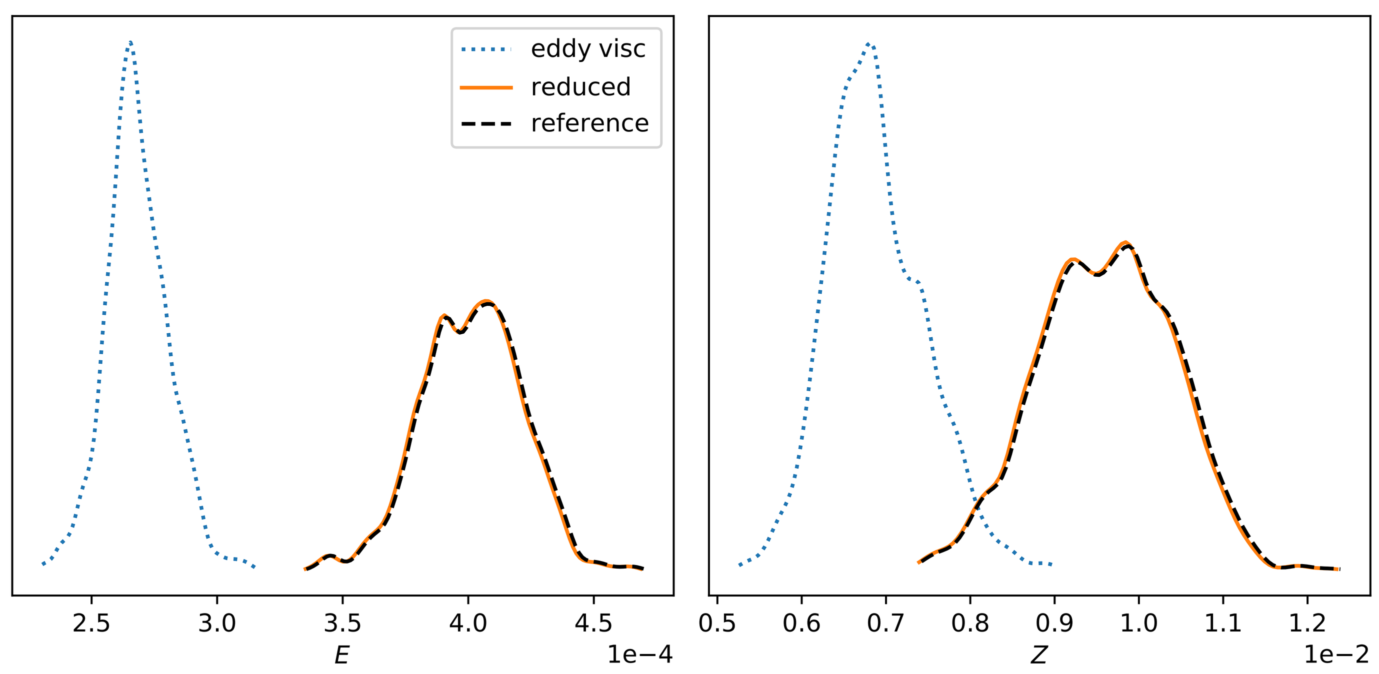
- Goal: $Q_i^{ref}(t) - Q_i(t) := \Delta Q_i$ is small $\forall t$ in training
- Simply equate SGS term to ΔQ_i:

$$\tau_i \left(\frac{\partial q_i}{\partial \omega}, P_i\right) = \Delta Q_i, \quad i = 1, \cdots, d$$

- imposes linear relaxation towards reference

6) Example results

- Track reference energy E and enstrophy Z ($d = 2$) $\rightarrow q_1 := \psi\omega/2$ and $q_2 = \omega\omega/2$



- Reduced SGS term yields the same E & Z pdf
- For every t, training data is reduced in size from 64^2 to 2

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