Approximation of Functions with Tensor Networks

Mazen Ali* and Anthony Nouy

Laboratoire de Mathématiques Jean Leray, Centrale Nantes, France

Introduction

Tensor Networks (TNs) are factorizations of high-dimensional tensors into products of low-order tensors. In the language of Neural Networks (NNs), some common TNs are instances of Convolutional Arithmetic Circuits, Sum-Product Networks and recurrent NNs (see [4]).

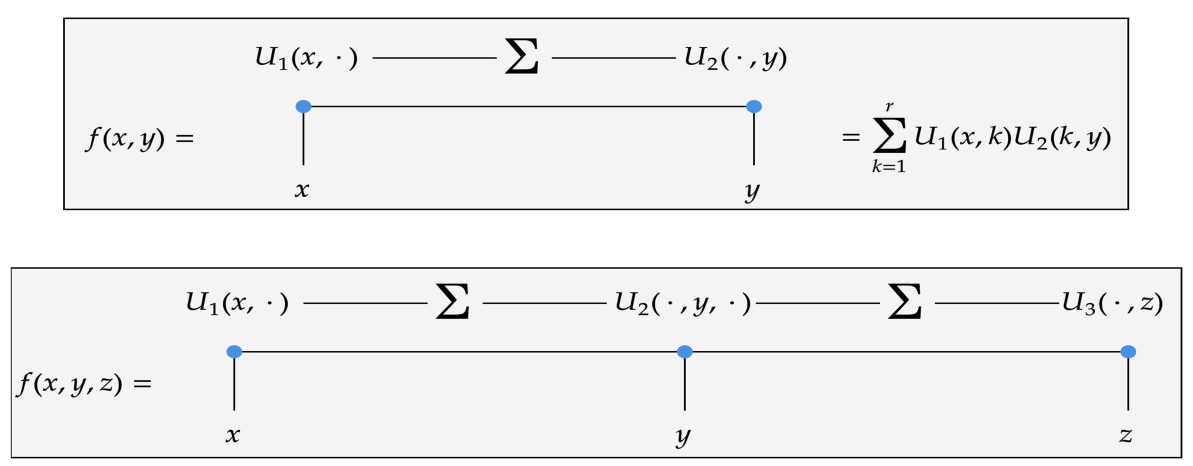
We study TNs as a tool for approximating functions. Our goal is to understand the nature of functions that can be efficiently approximated with TNs – referred to as *TN-function class* from now on. To that end, we (partially) answer two fundamental questions.

(i) Relation to known function classes: are functions in classical smoothness spaces also in the TN-function class?

(ii) Inherent structure: what can be said about the properties of TN-function classes? Are these functions themselves necessarily smooth in a classical sense?

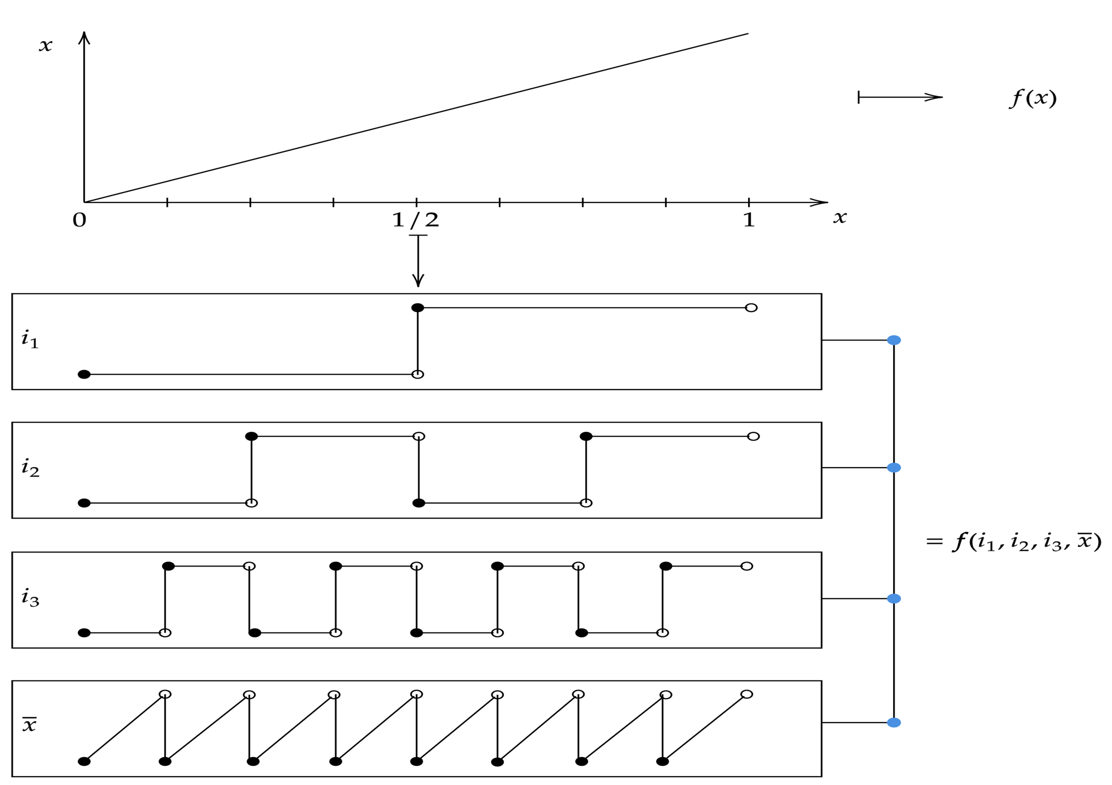
Tensor Networks

Multivariate Functions as TNs



Univariate Functions as TNs

$$[0,1) \ni x = \overbrace{i_1}^{\in\{0,1\}} 2^{-1} + \ldots + \overbrace{i_d}^{\in\{0,1\}} 2^{-d} + \overbrace{\bar{x}}^{\in[0,1)} 2^{-d}$$
$$f(x) = \boldsymbol{f}(i_1, \ldots, i_d, \bar{x})$$



"Zooming-in" into different pieces of the function. Highly nonlinear feature maps.

Results

Notation We measure the complexity of a TN-approximation by the total number of weights in the TN-network.

$$\begin{aligned} TN^\alpha &:= \text{"Functions in } L^p \text{ that can be approximated by TNs with rate } \alpha > 0\text{"} \\ W^\alpha &:= \text{"Fractional Sobolev space of smoothness } \alpha > 0\text{"} \\ B^\alpha &:= \text{"Besov space with smoothness } \alpha > 0, \\ &\quad\ \text{measured in the weaker } L^\tau\text{-norm with } 1/\tau = \alpha + 1/p\text{"} \end{aligned}$$

Direct Results

- For certain network topologies, TN^α is a Banach space.
- The relationship between TN^α, W^α and B^α is depicted in Figure 1.
- Analytic functions can be approximated with exponential rate of convergence.
- Similar results for multivariate approximation but with interesting nuances.

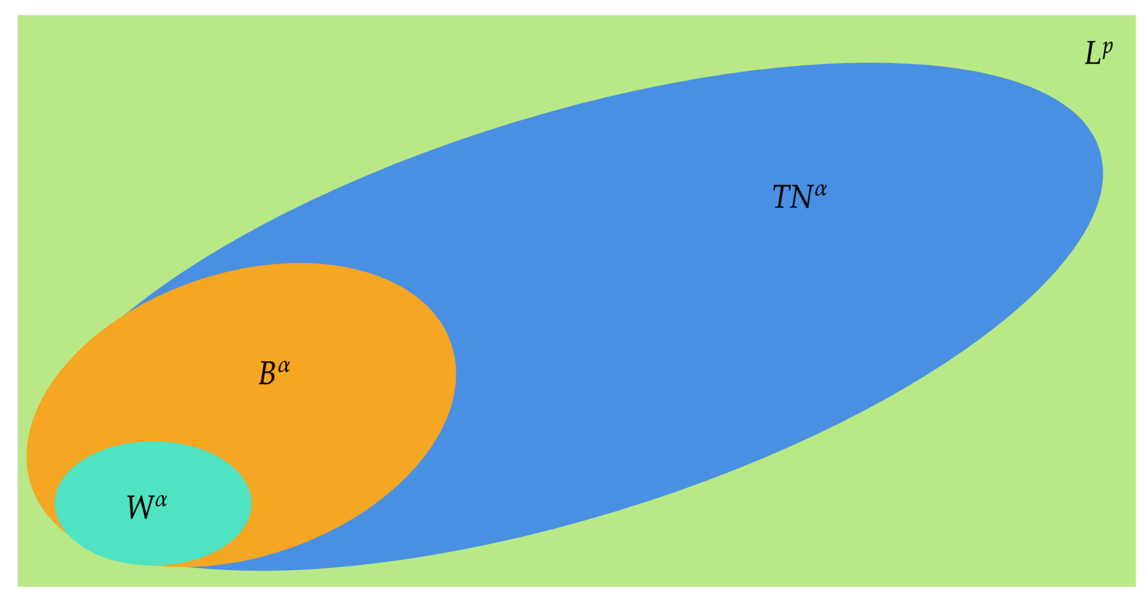


Fig. 1: Direct Embeddings for TN^α.

Inverse Results

- TN^α is not embedded in B^μ for *any* $\mu > 0$.
- TN complexity = depth d (resolution) + ranks $\Rightarrow$ two extremal regimes: high depth + small ranks (deep and narrow) OR small depth + high ranks (shallow and wide).
- Lack of regularity is due to high-depth-regime $\Rightarrow$ if one restricts to 2nd regime only, TN^α has some (small) Besov regularity $\mu > 0$, Figure 2.

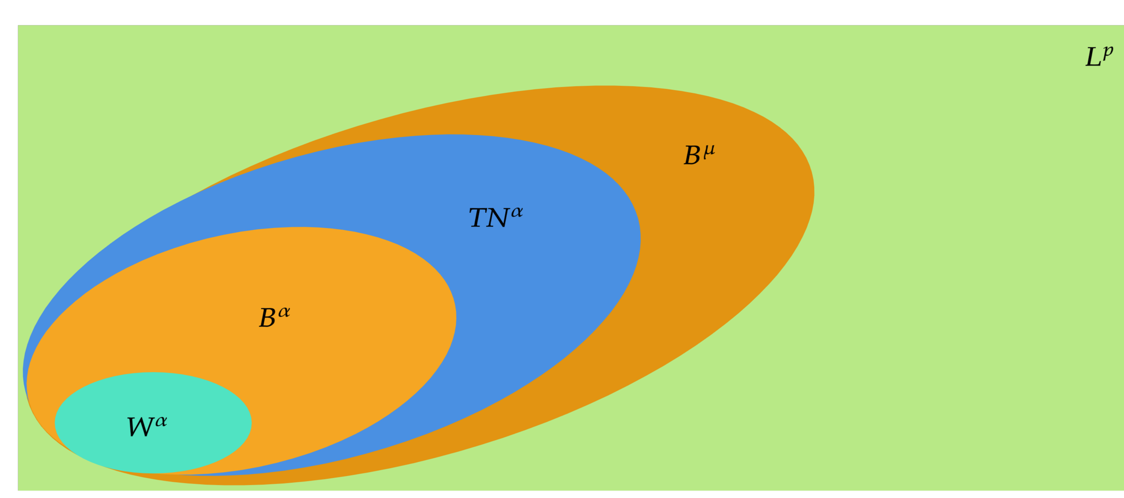


Fig. 2: Inverse Embedding of restricted $(TN_{\mathbb{R}})^\alpha$.

Summary

What we now understand

- TNs are highly efficient in representing exponentials, trigonometric polynomials, piece-wise polynomials, multi-resolution analysis, as well as more exotic and high-dimensional functions.
- TN-approximation can optimally replicate h-uniform, h-adaptive and hp-uniform/adaptive approximation.

What we still do not understand

- For classical function spaces as above, the network topology does not play a significant role. However, in "practice" it is known the network type is important and the optimal network is problem-dependent.
- What are the functions that can be approximated with networks of a specific topology? Do they have a simple description?
- Our results are restricted to tree tensor networks (without loops). Can we characterize the effect of loops?
- What is the effect of (stronger) nonlinearities?

More details can be found in [1, 2, 3].

Contact

- e-mail: mazen.ali@ec-nantes.fr
- web-page: `https://ali-mazen.com/`

Acknowledgements

The authors acknowledge AIRBUS Group for the financial support with the project AtRandom.

References

[1] Mazen Ali and Anthony Nouy. *Approximation with Tensor Networks. Part I: Approximation Spaces.* 2020. arXiv: 2007.00118 [math.FA].

[2] Mazen Ali and Anthony Nouy. *Approximation with Tensor Networks. Part II: Approximation Rates for Smoothness Classes.* 2020. arXiv: 2007.00128 [math.FA].

[3] Mazen Ali and Anthony Nouy. *Approximation with Tensor Networks. Part III: Approximation of Multi-Dimensional Functions.* In preparation.

[4] Nadav Cohen, Or Sharir, and Amnon Shashua. "On the Expressive Power of Deep Learning: A Tensor Analysis". In: *29th Annual Conference on Learning Theory.* Ed. by Vitaly Feldman, Alexander Rakhlin, and Ohad Shamir. Vol. 49. Proceedings of Machine Learning Research. Columbia University, New York, New York, USA: PMLR, 23–26 Jun 2016, pp. 698–728. URL: http://proceedings.mlr.press/v49/cohen16.html.